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Analysis facilis aequationem Riccatianam per fractionem continuam resolvendi

Leonhard Euler

INTEGRATIO GENERALIS AEQUATIONUM DIFFERENTIALIUM LINEARIUM CUJUSCUNQUE GRADUS

ET QUOTCUNQUE VARIABILES INVOLVENTIUM.

AUCTORE

L. EULERO.

Conventui exhib. die 28 Octobris 1779.

§. 1. Si fuerit functio quotcunque variabilium z, y, x, u, etc. determinanda ex aequatione differentiali cujuscunque gradus, in cujus singulis terminis quantitas V, cum suis differentialibus, unicam obtineat dimensionem, atque insuper coëfficientes singulorum terminorum fuerint constantes, tales aequationes hic voco lineares, et quemadmodum eas integrari oporteat, investigabo.

§. 2. Forma completa talium aequationum primo continebit ipsam quantitatem quaesitam V. Deinde occurrent differentialia primi gradus, quae sunt $\left(\frac{\partial V}{\partial z}\right)$, $\left(\frac{\partial V}{\partial y}\right)$, $\left(\frac{\partial V}{\partial x}\right)$, etc. quorum ergo numerus est $= n$, siquidem n fuerit numerus variabilium z, y, x, u, etc. Sequuntur termini differentiales secundi gradus: $\left(\frac{\partial \partial V}{\partial z^2}\right)$, $\left(\frac{\partial \partial V}{\partial z \partial y}\right)$, $\left(\frac{\partial \partial V}{\partial y^2}\right)$, $\left(\frac{\partial \partial V}{\partial z \partial x}\right)$, etc. quorum numerus est $\frac{n(n+1)}{1.2}$. Terminorum porro differentialium tertii gradus: $\left(\frac{\partial^3 V}{\partial z^3}\right)$; $\left(\frac{\partial^3 V}{\partial z^2 \partial y}\right)$; $\left(\frac{\partial^3 V}{\partial z \partial y^2}\right)$; $\left(\frac{\partial^3 V}{\partial y^3}\right)$; $\left(\frac{\partial^3 V}{\partial y^2 \partial x}\right)$; etc. numerus est $\frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$; sicque porro. Atque hi singuli termini, per quantitates constantes multiplicati, exhibebunt generatim aequationem differentialem linearem cujuscunque gradus, cujus integrationem hic accuratius sum perscrutaturus.

§. 3. Tales ergo aequationes omnes in hac forma generali continebuntur:

$$
\begin{aligned}
o = & A V + B\left(\frac{\partial V}{\partial z}\right) + C\left(\frac{\partial V}{\partial y}\right) + D\left(\frac{\partial V}{\partial x}\right) + \text{etc.} \\
& + E\left(\frac{\partial \partial V}{\partial z^2}\right) + F\left(\frac{\partial \partial V}{\partial y^2}\right) + G\left(\frac{\partial \partial V}{\partial x^2}\right) + H\left(\frac{\partial \partial V}{\partial z \partial y}\right) + I\left(\frac{\partial \partial V}{\partial z \partial x}\right) \\
& \qquad + K\left(\frac{\partial \partial V}{\partial x \partial y}\right) + \text{etc.} \\
& + L\left(\frac{\partial^3 V}{\partial z^3}\right) + M\left(\frac{\partial^3 V}{\partial y^3}\right) + \text{etc.} \\
& \qquad \text{etc.}
\end{aligned}
$$

Huic autem formae satis constat satisfacere talem formam integralem: $V = e^{\alpha z + \beta y + \gamma x + \text{etc.}}$. Hinc enim erit:

$$
\begin{aligned}
\left(\frac{\partial V}{\partial z}\right) &= \alpha e \cdots; & \left(\frac{\partial V}{\partial y}\right) &= \beta e \cdots; \\
\left(\frac{\partial V}{\partial x}\right) &= \gamma e \cdots; & \left(\frac{\partial \partial V}{\partial z^2}\right) &= \alpha \alpha e \cdots; \\
\left(\frac{\partial \partial V}{\partial y^2}\right) &= \beta \beta e \cdots; & \left(\frac{\partial \partial V}{\partial x^2}\right) &= \gamma \gamma e \cdots; \\
\left(\frac{\partial \partial V}{\partial z \partial y}\right) &= \alpha \beta e \cdots; & \left(\frac{\partial \partial V}{\partial z \partial x}\right) &= \alpha \gamma e \cdots; \\
\left(\frac{\partial \partial V}{\partial x \partial y}\right) &= \beta \gamma e \cdots; & \left(\frac{\partial^3 V}{\partial z^3}\right) &= \alpha^3 e \cdots; \\
\left(\frac{\partial^3 V}{\partial y^3}\right) &= \beta^3 e \cdots; & \left(\frac{\partial^3 V}{\partial x^3}\right) &= \gamma^3 e \cdots;
\end{aligned}
$$

et ita porro; quibus substitutis, quia totam aequationem per $e^{\alpha z + \beta y + \gamma x \text{ etc.}}$ dividere licet, patet, litteras assumtas α, β, γ, etc. hac aequatione determinari debere:

$$0 = A + B\alpha + C\beta + D\gamma + \text{ etc.}$$
$$+ E\alpha^2 + F\beta^2 + G\gamma^2 + H\alpha\beta + I\alpha\gamma + K\beta\gamma + \text{etc.}$$
$$+ L\alpha^3 + M\beta^3 + N\gamma^3 + \text{ etc.}$$
$$\text{etc.}$$

§. 4. Ex hac igitur aequatione, quam aequationis differentialis propositae *vicariam* appellare liceat, litterarum assumtarum α, β, γ, etc. quaelibet per reliquas, scilicet α per β, γ, etc. definiri poterit, idque tot modis, quoti gradus differentialis fuerit aequatio, ita ut hic reliquae litterae β, γ, etc. prorsus ab arbitrio nostro pendeant. Sic igitur singuli valores his litteris tributi praebebunt formulam determinatam $e^{\alpha x + \beta y + \gamma x + etc.}$, cujusmodi ergo formularum numerus prorsus erit infinitus, atque adeo eo altioris gradus, quo major fuerit numerus litterarum β, γ, etc. per quas primam α determinaverimus. Quod si ergo singulae istae formulae per coëfficientes constantes arbitrarios multiplicentur et in unam summam aggregentur, habebitur expressio maxime generalis, valorem quaesitum V exhibens, quam igitur tanquam integrale completum spectare licebit.

§. 5. Verum talis expressio, ob terminorum numerum infinities infinitum, hoc laborat incommodo, quod inde verus integralis valor perspici nequeat, quapropter porro erit investigandum, utrum talis expressio, in infinitum excurrens, non per certam formulam, seu functionem finitam, repraesentari possit, hocque commode succedet, quoties relationem inter litteras α, β, γ, etc. tali simplici formula $a\alpha + b\beta + c\gamma +$ etc. . . . $+ k = 0$ exhibere licet, id quod evenit quando ista formula fuerit factor formae vicariae supra allatae (§. 3.); tum enim omnes illas formulas exponentiales, numero infinitas, per certas functiones repraesentare licebit, quemadmodum in sequentibus ostendemus.

§. 6. Cum forma vicaria ex ipsa aequatione proposita sit formata, evidens est quemadmodum vicissim ipsa aequatio ex forma vicaria derivari possit. Si enim formula: $k + a\alpha + b\beta + c\gamma. +$ etc. fuerit factor formae vicariae, illi respondebit haec aequatio differentialis primi gradus: $kV + a\left(\frac{\partial V}{\partial z}\right) + b\left(\frac{\partial V}{\partial y}\right) + c\left(\frac{\partial V}{\partial x}\right) +$ etc. $= 0$, cujus ergo integrale simul erit quoque integrale ipsius aequationis propositae. Ad hoc igitur investigandum statuamus in genere esse $\partial V = p\partial z + q\partial y + r\partial x +$ etc. ut ista aequatio in hanc abeat:

$$kV = ap + bq + cr + \text{etc.} = 0.$$

§. 7. Ex hac jam aequatione quaeramus valorem ipsius $p = -\frac{kV}{a} - \frac{bq}{a} - \frac{cr}{a} -$ etc., qui in illa formula assumta substitutus dabit:

$$\partial V = -\frac{kV\partial z}{a} + q\left(\partial y - \frac{b\partial z}{a}\right) + r\left(\partial x - \frac{c\partial z}{a}\right),$$

quae aequatio hoc modo repraesentetur:

$$\partial V + \frac{kV\partial z}{a} = q\left(\partial y - \frac{b\partial z}{a}\right) + r\left(\partial x - \frac{c\partial z}{a}\right),$$

cujus membrum sinistrum integrabile manifesto redditur, si ducatur in $e^{\frac{kz}{a}}$, siquidem ejus integrale erit $e^{\frac{kz}{a}}V$, sicque nostra aequatio ita se habebit:

$$\partial . e^{\frac{kz}{a}} V = e^{\frac{kz}{a}} \times q\left(\partial y - \frac{b\partial z}{a}\right) + e^{\frac{kz}{a}} \times r\left(\partial x - \frac{c\partial z}{a}\right).$$

Haud difficulter autem intelligitur, quantitates q et r semper ita accipi posse, ut membrum dextrum etiam integrationem admittat.

§. 8. Quod quo facilius appareat, statuamus $y - \frac{bz}{a} = s$ et $x - \frac{cz}{a} = t$, fietque nostra aequatio:

$$\partial . e^{\frac{kz}{a}} \times V = e^{\frac{kz}{a}} \times (q\partial s + r\partial t),$$

ubi membrum postremum in genere refert differentiale functionis cujuscunque binarum variabilium s et t, unde colligitur integrando formulam $e^{\frac{kz}{a}}V$ aequari functioni cuicunque binarum variabilium s et t, quam more jam recepto hoc modo repraesentemus: $\Gamma : (s, t)$; ergo loco s et t restitutis valoribus orietur iste valor:

$$V = e^{-\frac{kz}{a}} \overline{\Gamma : \left(y - \frac{bz}{a}\right), \left(x - \frac{cz}{a}\right)}.$$

Hic scilicet valor aequationi propositae convenit, quoties ejus forma vicaria factorem habuerit $k + a\alpha + b\beta + c\gamma +$ etc.

§. 9. Quodsi forma vicaria praeterea alium habeat factorem simplicem, qui sit $k' + a'\alpha + b'\beta + c'\gamma +$ etc.

ex eo simili modo deducetur valor pro littera V, qui erit

$$V = e^{-\frac{k'z}{a'}} \Delta : \overline{\left(y - \frac{b'z}{a'}\right), \left(x - \frac{c'z}{a'}\right)},$$

qui cum praecedente quomodocunque conjungi poterit. Atque si forma vicaria in meros factores simplices, numero n, resolvi se patiatur, qui sint $k + a\alpha + b\beta + c\gamma +$ etc.; $k' + a'\alpha + b'\beta + c'\gamma +$ etc.; $k'' + a''\alpha + b''\beta + c''\gamma +$ etc.; etc., tum adeo integrale completum quantitatis V assignare poterimus, quod erit:

$$V = e^{-\frac{kz}{a}} \Gamma : \overline{\left(y - \frac{bz}{a}\right), \left(x - \frac{cz}{a}\right)} + e^{-\frac{k'z}{a'}} \Delta : \overline{\left(y - \frac{b'z}{a'}\right), \left(x - \frac{c'z}{a'}\right)}$$
$$+ e^{-\frac{k''z}{a''}} \Sigma : \overline{\left(y - \frac{b''z}{a''}\right), \left(x - \frac{c''z}{a''}\right)} + \text{etc.}$$

ubi characteres Γ; Δ; Σ; etc. denotant functiones quascunque arbitrarias quantitatum subnexarum.

§. 10. Verum eadem integralia, per functiones expressa, ex formula initio assumta $V = e^{\alpha z + \beta y + \gamma x}$ derivari possunt. Cum enim factor formae nostrae vicariae: $k + a\alpha + b\beta + c\gamma +$ etc. $= 0$ praebeat $\alpha = -\frac{k}{a} - \frac{b\beta}{a} - \frac{c\gamma}{a} -$ etc., exponens ipsius e erit $-\frac{kz}{a} + \beta\left(y - \frac{bz}{a}\right) + \gamma\left(x - \frac{cz}{a}\right)$, qui, posito ut ante brevitatis gratia $y - \frac{bz}{a} = s$ et $x - \frac{cz}{a} = t$, erit $-\frac{kz}{a} + \beta s + \gamma t$, sicque habebitur:

$$V = e^{-\frac{kz}{a} + \beta s + \gamma t} = e^{-\frac{kz}{a}} \cdot e^{\beta s} \cdot e^{\gamma t};$$

ubi probe notandum est litteras β et γ omnes possibiles valores recipere, ita ut $e^{\beta s}$ complectatur summam omnium ejus valorum, qui ex littera β oriri possunt, atque adeo omnes isti valores per quantitates constantes quascunque multiplicati sunt intelligendi. Aggregatum igitur omnium horum infinitorum valorum per $\int e^{\beta s}$ designemus. Similique modo $\int e^{\gamma t}$ omnes valores possibiles, qui ex variatione litterae γ nasci possunt, complectatur.

§. 11. Jam haud difficulter intelligitur, istam formam $\int e^{\beta s}$ omnes plane functiones ipsius s exhibere posse. Quod quo clarius appareat ponamus $s = lp$, fietque $e^{\beta s} = p^{\beta}$ et $\int e^{\beta s} = \int p^{\beta}$. Notum autem est, omnes functiones ipsius p resolvi posse in series, quarum termini procedant secundum potestates ipsius p, sicque formula $\int p^{\beta}$ complectitur omnes valores possibiles ipsius p, ideoque etiam omnes functiones ipsius s, quas ergo hoc charactere $\Gamma : s$ repraesentare licet. Simili modo, posito $t = lq$, patet $\int e^{\gamma t}$ aequivalere huic functioni: $\Delta : t$.

§. 12. Praeterea, quia singulas partes utriusque functionis in se multiplicare licet, manifestum est formulam $e^{\beta s + \gamma t} + 1$ non solum productum functionis ipsius s et functionis ipsius t indicare, sed etiam omnes plane functiones utcunque ex binis quantitatibus s et t formatas involvere, quam ergo expressionem hoc charactere: $\Gamma : \overline{s, t}$ repraesentamus. Hinc igitur, cum sit $s = y - \frac{bz}{a}$ et

$t = x - \frac{cz}{a}$, erit, uti ante per integrationem collegimus:

$$V = e^{-\frac{kz}{a}} \Gamma : \overline{(y - \tfrac{bz}{a}), (x - \tfrac{cz}{a})}.$$

§. 13. Hoc autem modo per functiones integralia hujusmodi aequationum differentialium exprimere non licet, nisi quatenus earum forma vicaria factores simplices comprehendit. Nisi enim hoc eveniat, integralia aliter exhiberi nequeunt, nisi omnia integralia particularia, quae ex formula $e^{\alpha z + \beta y + \gamma x + etc.}$ oriuntur, in unam summam colligendo. Quod quo clarius appareat, consideremus istum casum specialem: $(\frac{\partial \partial V}{\partial z^2}) = (\frac{\partial \partial V}{\partial x \partial y})$, cujus forma vicaria est $\alpha\alpha = \beta\gamma$, quae certe in factores simplices nullo modo resolvi potest. Hinc autem fit $\alpha = \sqrt{\beta\gamma}$, ideoque $V = e^{z\sqrt{\beta\gamma} + \beta y + \gamma x}$, ubi binis litteris β et γ omnes possibiles valores tribui sunt censendae, quas autem nullo modo sub quapiam functione definita complecti licet. Quo hoc clarius appareat ponamus $z = lp$, $y = lq$ et $x = lr$, ut fiat $V = p^{\sqrt{\beta\gamma}} q^{\beta} r^{\gamma}$. Quod si jam hic litteris α, β, γ, tantum valores integros tribuantur, prodibit talis aequatio:

$$V = \mathfrak{A} pqr + \mathfrak{B} p^{\sqrt{2}} q^2 r + \mathfrak{C} p^{\sqrt{3}} q^3 r + \text{etc.}$$

quos diversos terminos sub nulla certa functione complecti licet.

§. 14. Interim tamen, si aequatio vicaria resolutionem in duos factores, etsi non simplices, admittat, tum integrale aequationis propositae ad integrationem duarum aequationum inferioris gradus reducitur. Quodsi enim aequatio differentialis proposita ascendat ad gradum differentialium $m = \mu + \nu$, ejusque forma vicaria habeat duos factores, alterum gradus μ, alterum vero gradus ν, si ex his vicissim formentur aequationes differentiales, altera ad gradum μ, altera ad gradum ν assurget. Ponamus ergo integrationem prioris praebere $V = P$, posterioris vero $V = Q$, atque manifestum est, ipsius aequationis propositae integrale fore $AP + BQ$; atque adeo, si illa integralia fuerint completa, etiam hoc erit completum.

§. 15. Quae hactenus sunt tradita, proprie quidem ad functiones trium variabilium z, y, x, sunt accomodata: facile autem intelligitur, eadem praecepta pariter locum obtinere tam pro paucioribus, quam pro pluribus variabilibus.